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# Pseudoeffect Algebras. I. Basic Properties

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As a noncommutative generalization of effect algebras, we introduce pseudoeffect algebras and list some of their basic properties. For the purpose of a structure theory, we further define several kinds of Riesz-like properties for pseudoeffect algebras and show how they are interrelated.

## INTRODUCTION

One possible approach to a better understanding of the quantum mechanical formalism is to examine physically meaningful first-order structures derived from Hilbert space. The structure that probably has been most intensively studied is the orthomodular lattice of closed subspaces of the standard Hilbert space. Although optimal results have been obtained, not much has been gained concerning the foundational problems.

Over the last 10 years or so, attention has moved to another aspect of Hilbert space; instead of closed subspaces, which correspond to projection operators, all the positive operators lying below the identity, called (quantum) effects, are taken into consideration. Several different first-order axiom systems have been introduced, modeled by structures, the ground set of which may be chosen as the set of effects; among these are the weak orthoalgebras (Guintini and Greuling, 1989), effect algebras (Foulis and Bennett, 1994), and D-posets (Kôpka and Chovanec, 1994). These three theories are equivalent in the sense that each of them possesses definitional extensions to include the others. The first two use a partial sum, the last a partial difference operation.

In this paper, we generalize the second mentioned structure; we examine properties of pseudoeffect algebras that basically arise from effect algebras by dropping commutativity.

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We are working toward a structure theory of pseudoeffect algebras. Unfortunately, not much is known in this respect even for effect algebras. But the situation becomes much better when one postulates a certain property that may be compared with the Riesz Decomposition Property for po-groups. It is then in fact possible to show that our algebra is representable by an interval of a (not necessarily Abelian) po-group. Similar work was done by Ravindran (1996) for effect algebras.

The paper is divided into two parts. In Part I, we give the axioms and basic properties of the new structure and also introduce five nonequivalent properties of Riesz type. If one among these is fulfilled, a pseudoeffect algebra is an interval of a po-group, and this is proved in Part II. Finally, we give necessary and sufficient conditions for a pseudoeffect to be a pseudo-MV algebra; from this, it is possible to re-prove that pseudo-MV algebras are intervals in $\ell$-groups.

## 1. PSEUDOEFFECT ALGEBRAS

To model algebraically the set of quantum effects, that is, the set of self-adjoint operators of a Hilbert space between zero and identity, the notion of an effect algebra was introduced in Foulis and Bennett (1994). We recall that this algebra is a structure $(E; +, 0, 1)$, where $+$ is a partial binary operation and 0 and 1 are constants, such that, for all $a, b, c \in E$; (EA1) $a + b$ is defined iff $b + a$ is defined, in which case these elements are equal; (EA2) $(a + b) + c$ is defined iff $a + (b + c)$ is defined, in which case these elements are equal; (EA3) for exactly one $d \in E$, we have $a + d = 1$; and (EA4) if $1 + a$ is defined, then $a = 0$. We shall generalize this type of structure by dropping commutativity, that is, giving up (EA1).

*Definition 1.1.* A structure $(E; +, 0, 1)$, where $+$ is a partial binary operation and 0 and 1 are constants, is called a *pseudoeffect algebra* if, for all $a, b, c \in E$, the following hold.

(E1) $a + b$ and $(a + b) + c$ exist if and only if $b + c$ and $a + (b + c)$ exist, and in this case, $(a + b) + c = a + (b + c)$.
(E2) There is exactly one $d \in E$ and exactly one $e \in E$ such that $a + d = e + a = 1$.
(E3) If $a + b$ exists, there are elements $d, e \in E$ such that $a + b = d + a = b + e$.
(E4) If $1 + a$ or $a + 1$ exists, then $a = 0$.

In view of (E2), we may define the two unary operations $^\sim$ and $^-$ by requiring for any $a \in E$

$$\text{(EC)} \qquad a + a^\sim = a^- + a = 1.$$

*Remark.* We may also, and we will occasionally in the sequel, consider a pseudoeffect algebra as a structure $(E; +, ^\sim, ^-, 0, 1)$, where $+$ is a partial binary operation,

$^\sim$, $^-$ are unary operations, and 0, 1 are constants, such that (E1)–(E4) and (EC) hold.

It is obvious how effect algebras are characterized among pseudoeffect algebras.

*Definition 1.2.* Let $(E; +, 0, 1)$ be a pseudoeffect algebras. We say that two elements $a$ and $b$ of $E$ *commute* if $a + b$ and $b + a$ both exist and are equal.

We say that $E$ is *commutative* if, for $a, b \in E$, $a + b$ is defined if and only if $b + a$ is defined, in which case $a + b = b + a$.

**Proposition 1.3.** *Let $(E; +, 0, 1)$ be a pseudoeffect algebra. Then $E$ is an effect algebra if and only if $E$ is commutative.*

In the sequel, by any equation to hold we mean that all sums that occur in it exist, and it holds.

Because of the law of associativity, (E1), we may denote finite sums of elements of a pseudoeffect algebra without brackets.

**Lemma 1.4.** *Let $(E; +, 0, 1)$ be a pseudoeffect algebra. For all $a, b, c \in E$ we have the following:*

(i) $a + 0 = 0 + a = a$ *(i.e.,* 0 *is a neutral element).*
(ii) $a + b = 0$ *implies* $a = b = 0$ *(positivity).*
(iii) $0^\sim = 0^- = 1, 1^\sim = 1^- = 0$.
(iv) $a^{\sim -} = a^{-\sim} = a$.
(v) $a + b = a + c$ *implies* $b = c$, *and* $b + a = c + a$ *implies* $b = c$ *(cancellation laws).*
(vi) $a + b = c$ *iff* $a = (b + c^\sim)^-$ *iff* $b = (c^- + a)^\sim$.

**Proof:** We will prove first (v), then (iv), (iii), (i), and (ii), and finally (vi).

(v) Suppose $a + b = a + c$. Then by (E2), for some $d$, we have $d + (a + b) = d + (a + c) = 1$, and by (E1), $(d + a) + b = (d + a) + c = 1$. By (E2), it follows that $b = (d + a)^\sim = c$.

Similarly, from $b + a = c + a$, it follows that $(b + a) + d = (c + a) + d = 1$ for some $d$, so $b + (a + d) = c + (a + d) = 1$ and $b = (a + d)^- = c$.

(iv) By (E2), we have $a + a^\sim = 1 = a^{\sim -} + a^\sim$, and so by (v), $a = a^{\sim -}$.

Similarly, $a^- + a = 1 = a^- + a^{-\sim}$, and so $a = a^{-\sim}$.

(iii) By (E2), we have $1 + 1^\sim = 1^- + 1 = 1$, and by (E4), $1^\sim = 1^- = 0$. By (iv), we get $1 = 1^{\sim -} = 0^-$ and $1 = 1^{-\sim} = 0^\sim$.

(i) By (iii) and (E1), we have $a^- + a = 1 = 1 + 1^\sim = 1 + 0 = (a^- + a) + 0 = a^- + (a + 0)$, and by (v), $a = a + 0$.

Similarly, $a + a^{\sim} = 1 = 1^{-} + 1 = 0 + 1 = (0 + a) + a^{\sim}$ and $a = 0 + a$.

(ii) By (i), $a + b = 0$ implies $b^{\sim} = a + b + b^{\sim} = a + 1$, which, by (E4), means $a = 0$, so by (iv), (i), and (iii), $b = b^{\sim -} = (0 + 1)^{-} = 1^{-} = 0$.

(vi) Suppose $c = a + b$. Then $a + b + c^{\sim} = 1 = a + a^{\sim}$, so $b + c^{\sim} = a^{\sim}$ and $a = a^{\sim -} = (b + c^{\sim})^{-}$. Similarly, $c^{-} + a + b = 1 = b^{-} + b$, so $b = (c^{-} + a)^{\sim}$. Suppose $a = (b + c^{\sim})^{-}$. Then $a^{\sim} = b + c^{\sim}$, so $1 = a + b + c^{\sim}$ and $c = a + b$. Suppose $b = (c^{-} + a)^{\sim}$. Then $b^{-} = c^{-} + a$, so $1 = c^{-} + a + b$ and $c = a + b$. $\square$

We introduce in the usual manner a partial order for pseudoeffect algebras.

*Definition 1.5.* Let $(E; +, 0, 1)$ be a pseudoeffect algebra. We define for $a, b \in E$

$$a \leq b \quad \text{iff } a + c = b \text{ for some } c \in E.$$

*Remark.* From (E3), it is clear that

$$a \leq b \quad \text{iff } d + a = b \text{ for some } d \in E,$$

or, in other words, our order is two-sided. This was in fact the main motivation for choosing the axiom (E3).

**Lemma 1.6.** *Let $(E; +, 0, 1)$ be a pseudoeffect algebra. The following hold in E for all $a, a_1, b, b_1, c \in E$:*

(i) *$\leq$ is a partial order on $E$.*
(ii) *$a \leq b$ iff $b^{-} \leq a^{-}$ iff $b^{\sim} \leq a^{\sim}$. That is, $^{-}$ and $^{\sim}$ are isomorphisms of the order of $E$ onto the dual order of $E$.*
(iii) *If $a + b$ exists, $a_1 \leq a$, and $b_1 \leq b$, then also $a_1 + b_1$ exists.*
(iv) *$a + b$ exists iff $a \leq b^{-}$ iff $b \leq a^{\sim}$.*
(v) *Suppose $b + c$ exists. Then $a \leq b$ if and only if $a + c$ exists and $a + c \leq b + c$. Suppose $c + b$ exists. Then $a \leq b$ if and only if $c + a$ exists and $c + a \leq c + b$.*

**Proof:**

(i) $a \leq a$, because $a + 0 = a$.

Now, $a \leq b$ and $b \leq a$ imply $a + a_1 = b$ and $b + b_1 = a$ for some $a_1$ and $b_1$, so $a + a_1 + b_1 = a = a + 0$; this means $a_1 + b_1 = 0$ by Lemma 1.4(v) and $a_1 = b_1 = 0$ by Lemma 1.4(ii); so $a = b$.

Now, $a \leq b$ and $b \leq c$ imply $a + a_1 = b$ and $b + b_1 = c$ for some $b_1$ and $c_1$, so $a + a_1 + b_1 = c$, which means $a \leq c$.

(ii) $a \leq b$ iff, for some $c$, $c + a = b$ iff, by Lemma 1.4(vi), for some $c$, $a^- = b^- + c$ iff $b^- \leq a^-$.

Similarly, $a \leq b$ iff, for some $c$, $a + c = b$ iff, for some $c$, $a^\sim = c + b^\sim$ iff $b^\sim \leq a^\sim$.

(iii) If $a + b$ exists and $a_1' + a_1 = a$ and $b_1 + b_1' = b$ for some $a_1'$ and $b_1'$, it follows that $(a_1' + a_1) + (b_1 + b_1')$ exists and so, by (E1), that $a_1 + b_1$ exists.

(iv) $a + b$ exists iff, for some $d$, $a + b + d = 1$ iff, for some $d$, $a^\sim = b + d$ iff $b \leq a^\sim$.

Similarly, $a + b$ exists iff, for some $d$, $d + a + b = 1$ iff, for some $d$, $b^- = d + a$ iff $a \leq b^-$.

(v) Suppose $b + c$ exists. Then $a \leq b$ iff, for some $d$, $d + a = b$ iff, for some $d$, $d + a + c = b + c$ iff $a + c$ exists and $a + c \leq b + c$.

Similarly, suppose that $c + b$ exists. Then $a \leq b$ iff, for some $d$, $a + d = b$ iff, for some $d$, $c + a + d = c + b$ iff $c + a$ exists and $c + a \leq c + b$. □

**Lemma 1.7.** *Let $(E; +, 0, 1)$ be a pseudoeffect algebra. For all $a, b, c \in E$, we have the following.*

(i) *Let $c + a$, $c + b$, and $(c + a) \wedge (c + b)$ exist. Then $a \wedge b$ and $c + (a \wedge b)$ exist, and we have $c + (a \wedge b) = (c + a) \wedge (c + b)$. Let $a + c$, $b + c$, and $(a + c) \wedge (b + c)$ exist. Then $a \wedge b$ and $(a \wedge b) + c$ exist, and we have $(a \wedge b) + c = (a + c) \wedge (b + c)$.*

(ii) *Let $a \vee b$ and $c + (a \vee b)$ exist. Then $c + a$, $c + b$, and $(c + a) \vee (c + b)$ exist, and we have $c + (a \vee b) = (c + a) \vee (c + b)$. Let $a \vee b$ and $(a \vee b) + c$ exist. Then $a + c$, $b + c$, and $(a + c) \vee (b + c)$ exist, and we have $(a \vee b) + c = (a + c) \vee (b + c)$.*

**Proof:**

(i) As $c \leq (c + a) \wedge (c + b)$, we have for some $d$ that $c + d = (c + a) \wedge (c + b)$. Then $c + d \leq c + a, c + b$, and by Lemma 1.6(v), $d \leq a, b$. Suppose $x \leq a, b$. Then again by Lemma 1.6(v), we have $c + x \leq c + a, c + b$, that is, $c + x \leq (c + a) \wedge (c + b) = c + d$ and $x \leq d$. It follows that $d = a \wedge b$.

The second part of (i) is proved similarly.

(ii) As $a, b \leq a \vee b$, we have by Lemma 1.6(v) that $c + a$ and $c + b$ exist and $c + (a \vee b) \geq c + a, c + b$. Let $x \geq c + a, c + b$. Then for some $y, z$, we have $x = c + a + y = c + b + z$. So $a + y = b + z \geq a \vee b$, and by Lemma 1.6(v), we get $x \geq c + (a \vee b)$. It follows that $c + (a \vee b) = (c + a) \vee (b + a)$.

The second part of (ii) is proved similarly. □

*Remark 1.8.* Lemma 1.7 may be easily generalized to the case that infinite infima or suprema occur rather than infima or suprema of pairs of elements. So, for example, if for $a_\iota, c \in E$, where $\iota$ runs over some index set, the term $\bigwedge_\iota(c + a_\iota)$ exists, we may prove similarly as above that $c + \bigwedge_\iota a_\iota = \bigwedge_\iota(c + a_\iota)$.

In the sequel, by a property of partially ordered sets to hold in a pseudoeffect algebra $E$, we mean that this property holds in $(E; \leq)$, where $\leq$ is the partial order of $E$ as introduced above. In particular, $E$ is said to be complete, $\sigma$-complete, or atomic iff $E$, considered as a partial ordered set, is complete, $\sigma$-complete, or atomic, respectively. Moreover, we call $E$ a lattice pseudoeffect algebra if $(E; \leq)$ is a lattice.

## 2. INTERVAL PSEUDOEFFECT ALGEBRAS

We are interested in pseudoeffect algebras that arise from intervals in partially ordered groups in the following manner.

*Definition 2.1.* Let $(G; +, \leq)$ be a po-group and $u$ a positive element of $G$.

(i) We denote by $(G, u)$ the structure $(G; +, \leq, u)$, that is, we add the element $u$ as a constant. $(G, u)$ is called a *unital po-group* if $u$ is a strong unit of $G$, that is, if for all $g \in G$, there is an $n \in \mathbb{N}$ such that $-nu \leq g \leq nu$.

(ii) We call the set

$$\Gamma(G, u) \stackrel{\text{def}}{=} \{g \in G^+ \colon g \leq u\}$$

the *unit interval* of $(G, u)$. We denote by $(\Gamma(G, u); +, 0, u)$ the structure consisting of the unit interval of $(G, u)$, the partial binary operation $+$ that is the restriction of the group addition to those pairs of elements of $\Gamma(G, u)$ whose sum lies again in $\Gamma(G, u)$, the neutral element of $G$, 0, and the positive element $u$.

As is easily checked, $(\Gamma(G, u); +, 0, u)$ is a pseudoeffect algebra. For $g \in \Gamma(G, u)$, we have here

$$g^{\sim} = -g + u,$$

$$g^{-} = u - g.$$

Furthermore, it is clear that the order defined for $(\Gamma(G, u); +, 0, u)$ coincides with the order of the po-group $G$ restricted to $\Gamma(G, u)$.

*Definition 2.2.*

(i) A pseudoeffect algebra $(E; +, 0, 1)$ is called an *interval pseudoeffect algebra*, if there is a unital po-group $(G, u)$ such that $(E; +, 0, 1)$ and $(\Gamma(G, u); +, 0, u)$ are isomorphic.

(ii) By a homomorphism from a pseudoeffect algebra $(E; +, 0, 1)$ into a unital po-group $(G; +, \leq, u)$, we mean a function $\varphi\colon E \to G$ such that $\varphi$ maps into the positive cone of $G$, the sum, whenever defined, is preserved, $\varphi(0)$ is the neutral element, and $\varphi(1) = u$.

An example of a noncommutative po-group leading to a noncommutative pseudoeffect algebra is the following (Darnel, 1995, Example 4.1).

**Example 2.3.** Let $G = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$; define for every two elements of $G$

$$(a_1, b_1, c_1) + (a_2, b_2, c_2) \stackrel{\text{def}}{=} \begin{cases} (a_1 + a_2, b_1 + b_2, c_1 + c_2) & \text{if } a_2 \text{ is even,} \\ (a_1 + a_2, b_2 + c_1, b_1 + c_2) & \text{if } a_2 \text{ is odd;} \end{cases}$$

and define $(a_1, b_1, c_1) \leq (a_2, b_2, c_2)$ to hold if $a_1 < a_2$ or $a_1 = a_2, b_1 \leq b_2$, and $c_1 \leq c_2$.

Then $(G; +, \leq)$ is an $\ell$-group, and

$$\Gamma(G, (1, 0, 0)) = \{(0, b, c)\colon b, c \geq 0\} \cup \{(1, b, c)\colon b, c \leq 0\}$$

becomes a pseudoeffect algebra with the sum and constants defined according to Definition 2.1.

Both structures are noncommutative because, $(0, 1, 2) + (1, -2, -2) = (1, 0, -1)$, but $(1, -2, -2) + (0, 1, 2) = (1, -1, 0)$.

## 3. PSEUDOEFFECT ALGEBRAS WITH RIESZ PROPERTIES

Our aim is to develop a structure theory for pseudoeffect algebras. But this is hardly possible in the general case. What we do here is what was proposed for effect algebras by Ravindran (1996); that is, we assume a property that is comparable to the Riesz Decomposition Property of po-groups.

Now there are several different possibilities for defining a property of Riesz type for pseudoeffect algebras. In the present section, we shall in fact introduce not less than five different ones. It is of interest that no one of these is equivalent to any of the others, but they are linearly orderable by strength.

*Definition 3.1.* Let $(E; +, 0, 1)$ be a pseudoeffect algebra.

(a) For $a, b \in E$, we write $a$ **com** $b$ to mean that for all $a_1 \leq a$ and $b_1 \leq b$, $a_1$ and $b_1$ commute.

(b) We say that $E$ fulfils the *Riesz Interpolation Property* (RIP) if, for any $a_1, a_2, b_1, b_2 \in E$ such that $a_1, a_2 \leq b_1, b_2$, there is a $c \in E$ such that $a_1, a_2 \leq c \leq b_1, b_2$.

(c) We say that $E$ fulfils the *Weak Riesz Decomposition Property* ($\mathrm{RDP}_0$) if, for any $a, b_1, b_2 \in E$ such that $a \leq b_1 + b_2$, there are $d_1, d_2 \in E$ such that $d_1 \leq b_1, d_2 \leq b_2$, and $a = d_1 + d_2$.
(d) We say that $E$ fulfils the *Riesz Decomposition Property* (RDP) if, for any $a_1, a_2, b_1, b_2 \in E$ such that $a_1 + a_2 = b_1 + b_2$, there are $d_1, d_2, d_3, d_4 \in E$ such that $d_1 + d_2 = a_1, d_3 + d_4 = a_2, d_1 + d_3 = b_1$, and $d_2 + d_4 = b_2$.
(e) We say that $E$ fulfils the *Commutational Riesz Decomposition Property* ($\mathrm{RDP}_1$) if, for any $a_1, a_2, b_1, b_2 \in E$ such that $a_1 + a_2 = b_1 + b_2$, there are $d_1, d_2, d_3, d_4 \in E$ such that (i) $d_1 + d_2 = a_1, d_3 + d_4 = a_2, d_1 + d_3 = b_1, d_2 + d_4 = b_2$ and (ii) $d_2$ **com** $d_3$.
(f) We say that $E$ fulfils the *Strong Riesz Decomposition Property* ($\mathrm{RDP}_2$) if, for any $a_1, a_2, b_1, b_2 \in E$ such that $a_1 + a_2 = b_1 + b_2$, there are $d_1, d_2, d_3, d_4 \in E$ such that (i) $d_1 + d_2 = a_1, d_3 + d_4 = a_2, d_1 + d_3 = b_1, d_2 + d_4 = b_2$ and (ii) $d_2 \wedge d_3 = 0$.

It is clear that the **com**-relation is symmetric.

We note further that from the Riesz Decomposition Property (RDP) already one special case of what is required in condition (ii) of ($\mathrm{RDP}_1$) easily follows: With respect to the notations used in (d) and (e), we have from (RDP) that $d_2$ and $d_3$ commute. Indeed, $d_1 + d_2 + d_3 + d_4 = a_1 + a_2 = b_1 + b_2 = d_1 + d_3 + d_2 + d_4$ and so, by Lemma 1.4(v), $d_2 + d_3 = d_3 + d_2$.

**Lemma 3.2.** *Let $(E; +, 0, 1)$ be a pseudoeffect algebra fulfilling ($\mathrm{RDP}_0$).*

(i) *Let $a, b, c \in E$ and $a + b$ exist. Then from $a$ **com** $c$ and $b$ **com** $c$ it follows that $a + b$ **com** $c$.*
(ii) *If $a \wedge b = 0$, then $a + b, b + a$, and $a \vee b$ exist and are all equal.*

**Proof:**

(i) Suppose $a$ **com** $c$ and $b$ **com** $c$ and $d \leq a + b, c_1 \leq c$. Because of ($\mathrm{RDP}_0$) there are elements $d_1$ and $d_2$ such that $d_1 \leq a, d_2 \leq b$, and $d = d_1 + d_2$. By assumption, each of $d_1$ and $d_2$ commutes with $c_1$; so also $d = d_1 + d_2$ commutes with $c_1$.
(ii) Let $c$ be given such that $c \geq a, b$; there is of course at least one such element. So, for some $x, c = x + b$, and by ($\mathrm{RDP}_0$), there are elements $d_1 \leq x, d_2 \leq b$ such that $a = d_1 + d_2$. Because $d_2 \leq a, b$, we have $d_2 = 0$, and so $a \leq x$. It follows that $c = x + b \geq a + b$. In particular, $a + b$ exists, and as $a + b$ is an upper bound of $a$ and $b$, it follows that $a + b = a \vee b$. $\square$

**Proposition 3.3.** *Let* $(E; +, 0, 1)$ *be a pseudoeffect algebra.*

(i) *We have the implications*

$$(RDP_2) \Rightarrow (RDP_1) \Rightarrow (RDP) \Rightarrow (RDP_0) \Rightarrow (RIP).$$

*The converse of any of these implications does not hold.*

(ii) *$E$ fulfils $(RDP_2)$ if and only if $E$ is lattice-ordered and fulfils $(RDP_0)$.*

(iii) *Let $E$ be commutative. Then we have the implications*

$$(RDP_2) \Rightarrow (RDP_1) \Leftrightarrow (RDP) \Leftrightarrow (RDP_0) \Rightarrow (RIP).$$

*Any implication not shown here does not hold.*

**Proof:**

(i) $(\mathrm{RDP}_2)$ implies $(\mathrm{RDP}_1)$ by Lemma 3.2(ii); $(\mathrm{RDP}_1)$ implies trivially (RDP); and it is evident that (RDP) implies $(\mathrm{RDP}_0)$.

Now suppose $(\mathrm{RDP}_0)$ and $a, b \leq c, d$. Then there is an $a_1 \in E$ such that $a + a_1 = c$, and from $b \leq a + a_1$ it follows by $(\mathrm{RDP}_0)$ that there are $e \leq a$ and $\overline{b} \leq a_1$ such that $b = e + \overline{b}$. Moreover, for some $\overline{a} \in E$, we have $a = e + \overline{a}$, and because $e \leq a, b$, for some $\overline{c}, \overline{d}$, also $c = e + \overline{c}$ and $d = e + \overline{d}$.

By Lemma 1.6(v), we have $\overline{a}, \overline{b} \leq \overline{c}, \overline{d}$, and from $e + \overline{a} + \overline{b} \leq a + a_1 = c = e + \overline{c}$, it follows that $\overline{a} + \overline{b} \leq \overline{c}$.

Now choose $b_1 \in E$ such that $b_1 + \overline{b} = \overline{d}$, and because $\overline{a} \leq b_1 + \overline{b}$, there are $\overline{\overline{a}} \leq b_1$ and $f \leq \overline{b}$ such that $\overline{a} = \overline{\overline{a}} + f$. Choose $\overline{\overline{b}} \in E$ such that $\overline{b} = \overline{\overline{b}} + f$. Let $\overline{x} = \overline{\overline{a}} + \overline{\overline{b}} + f$. Then $\overline{a}, \overline{b} \leq \overline{x} \leq \overline{a} + \overline{b} \leq \overline{c}$ as well as $\overline{x} \leq b_1 + \overline{b} = \overline{d}$. So $x = e + \overline{x}$ is the interpolant required to show that (RIP) holds.

The examples given later show that the converse of any of the implications does not hold.

(ii) Let $E$ fulfil $(\mathrm{RDP}_2)$. By (i), $E$ fulfils $(\mathrm{RDP}_0)$.

Let $a, b \in E$. Then from $a + a^{\sim} = b + b^{\sim}$, it follows that there are elements $d_1, d_2, d_3, d_4$ such that $d_1 + d_2 = a$, $d_3 + d_4 = a^{\sim}$, $d_1 + d_3 = b$, $d_2 + d_4 = b^{\sim}$, and $d_2 \wedge d_3 = 0$. Then according to Lemma 3.2(ii), we have $d_2 + d_3 = d_2 \vee d_3$, and according to Lemma 1.7(ii), we have $d_1 + d_2 + d_3 = d_1 + (d_2 \vee d_3) = (d_1 + d_2) \vee (d_1 + d_3) = a \vee b$. So all suprema exist.

It is easy to see that $(a^{\sim} \vee b^{\sim})^{-} = a \wedge b$. So also all infima exist. Hence $E$ is lattice-ordered.

Let $E$ be lattice-ordered and fulfil $(\mathrm{RDP}_0)$. Let $a_1 + a_2 = b_1 + b_2$. Set $d_1 = a_1 \wedge b_1$ and $d_2, d_3$ in such a way that $a_1 = d_1 + d_2$, $b_1 = d_1 + d_3$.

We claim that $d_2 \leq b_2$. Indeed, from $a_1 \leq b_1 + b_2$, we get by (RDP$_0$) that $a_1 = e_1 + e_2$ for some $e_1 \leq b_1, e_2 \leq b_2$, and from $e_1 \leq a_1 \wedge b_1 = d_1$, we have $a_1 \leq d_1 + b_2$, so we conclude $d_2 \leq b_2$. Choose $d_4 \in E$ such that $d_2 + d_4 = b_2$. Similarly, we may choose $d_4' \in E$ such that $d_3 + d_4' = a_2$.

Furthermore, $d_2 \wedge d_3 = 0$. For, by Lemma 1.7(i), we have $d_1 = a_1 \wedge b_1 = (d_1 + d_2) \wedge (d_1 + d_3) = d_1 + (d_2 \wedge d_3)$. From this we conclude that $d_1 + d_2 + d_3 + d_4' = a_1 + a_2 = b_1 + b_2 = d_1 + d_3 + d_2 + d_4 = d_1 + d_2 + d_3 + d_4$, so $d_4' = d_4$. So $d_1, d_2, d_3, d_4$ fulfil the requirements of Definition 3.1(f) of (RDP$_2$).

(iii) (RDP$_1$) trivially follows from (RDP). (RDP) is to be derived from (RDP$_0$) in the obvious manner (Ravindran, 1996, Lemma 2.12). The other implications are proved in (i).
The examples given later show that none of the implications not shown holds. □

For every possible combination of the different kinds of Riesz properties to hold or not to hold in a pseudoeffect algebra, we will now give one example.

As an example of a pseudoeffect algebra in which (RDP$_2$) holds, a unit interval in any $\ell$-group may serve (see, e.g., Fuchs, 1963, Theorem V.1). In this way, a noncommutative pseudoeffect algebra has already been constructed; see Example 2.3.

**Example 3.4.** (RDP$_1$) $\not\Rightarrow$ (RDP$_2$). Let $E$ be the set of rational functions from the real unit interval to itself such that no singularities occur. Define $+$ to be the pointwise addition of two such functions whenever this leads to a result within $E$; let 0 and 1 be the constant functions with value 0 and 1, respectively. Then $(E; +, 0, 1)$ is obviously an effect algebra and a fortiori a pseudoeffect algebra. We shall see that (RDP$_1$) holds in $E$, but (RDP$_2$) does not.

If, for $f_1, f_2, g_1, g_2 \in E$, the equation $f_1 + f_2 = g_1 + g_2$ holds, then the four, continuously extended whenever necessary, functions $f_1 g_1/(g_1 + g_2)$, $f_1 g_2/(g_1 + g_2)$, $f_2 g_1/(g_1 + g_2)$, and $f_2 g_2/(g_1 + g_2)$ obviously fulfil the requirements of the definition of (RDP). So, by commutativity, (RDP$_1$) holds in $E$.

On the other hand, (RDP$_2$) would by Proposition 3.3(ii) force an infimum of any two functions of $E$ to exist in $E$, which is not the case. So (RDP$_2$) fails to hold.

**Example 3.5.** (variation of Fuchs, 1965, Example 3.10) (RDP) $\not\Rightarrow$ (RDP$_1$). Let $G$ be an additive group generated by the countably many elements $g_0, g_1, \ldots$; let $v\colon (G; +) \to (\mathbb{R}, +)$ be the homomorphism determined by the conditions

$v(g_i) = (\frac{1}{2})^i$, $i = 0, \ldots$; and let $G$ fulfil the condition that every $a \in G$ such that $v(a) = 0$ commutes with any other $b \in G$. Define a partial order in $G$ by setting $G^+ \stackrel{\text{def}}{=} \{x \in G : x = 0 \text{ or } v(x) > 0\}$; this means that we have for $a, b \in G$

$$a \leq b \quad \text{iff } a = b \text{ or } v(a) < v(b). \tag{1}$$

We shall see that $\Gamma(G, g_0)$ fulfils (RDP), but not ($\text{RDP}_1$).

To prove (RDP), let $a_1, a_2, b_1, b_2 \in \Gamma(G, g_0)$ be such that $a_1 + a_2 = b_1 + b_2$; we may suppose that $a_1, a_2, b_1, b_2 > 0$. We have to show that for some $k \in G$, the scheme

$$\begin{array}{ccc} a_1 - k & k & \rightarrow \; a_1 \\ k - a_1 + b_1 & -k + b_2 & \rightarrow \; a_2 \\ \downarrow & \downarrow & \\ b_1 & b_2 & \end{array} \tag{2}$$

holds and every element in it is in $\Gamma(G, g_0)$.

If $v(a_1) < v(b_1)$, we put $k = 0$; if $v(a_1) > v(b_1)$, we put $k = -b_1 + a_1$. Let now $v(a_1) = v(b_1)$ and, because the case $a_1 = b_1$ is trivial, let $a_1 \neq b_1$. Because then, $-a_1 + b_1$ commutes with $k$ however chosen, we may put $k = g_i$ choosing $i$ large enough to make every element in (2) strictly positive. So (RDP) is proved.

Now consider again an equation $a_1 + a_2 = b_1 + b_2$, where $0 < a_1, a_2, b_1, b_2 \leq g_0$, $v(a_1) = v(b_1)$, and $a_1 \neq b_1$. For (2) to hold means that $k$ can be neither 0 nor $-b_1 + a_1$. It follows that for a sufficiently large natural number $i$, we have $g_i \leq k - a_1 + b_1$ and $g_{i+1} \leq k$; but $g_i$ and $g_{i+1}$ do not commute. So ($\text{RDP}_1$) does not hold.

**Example 3.6.** ($\text{RDP}_0$) $\not\Rightarrow$ (RDP). Similar to the previous example, let $G$ be the additive group generated freely by countably many elements $g_0, g_1, \ldots$; let $v : (G; +) \rightarrow (\mathbb{R}, +)$ be the homomorphism determined by the conditions $v(g_{2i}) = v(g_{2i+1}) = (\frac{1}{2})^i$, $i = 0, \ldots$. Define a partial order in $G$ by setting $G^+ \stackrel{\text{def}}{=} \{x \in G : x = 0 \text{ or } v(x) > 0\}$; then again (1) holds. We shall see that $\Gamma(G, g_0)$ fulfils ($\text{RDP}_0$), but not (RDP).

Indeed, let $a, b_1, b_2 \in \Gamma(G, g_0)$ be such that $a \leq b_1 + b_2$. By (1), it is then clear that for some $k \in \Gamma(G, g_0)$, we have $0 \leq a - k \leq b_1$ and $0 \leq k \leq b_2$. So ($\text{RDP}_0$) holds.

On the other hand, consider the equation $g_2 + (-g_2 + 3g_4) = g_3 + (-g_3 + 3g_4)$, and suppose that there are four elements as required by Definition 3.1(d).

This means that for some $k \in G$ the following scheme holds:

$$\begin{array}{ccccc} g_2 - k & & k & \rightarrow & g_2 \\ \overset{k-g_2+g_3}{\underset{-g_2+g_3+k}{=}} & & -k - g_3 + 3g_4 & \rightarrow & -g_2 + 3g_4 \\ \downarrow & & \downarrow & & \\ g_3 & & -g_3 + 3g_4. & & \end{array} \tag{3}$$

We have especially $k + (-g_2 + g_3) = (-g_2 + g_3) + k$. From this it follows that $k = z(-g_2 + g_3)$ for some $z \in \mathbb{Z}$. Now $k$ is required to be in $G^+$, which is only the case for $z = 0$. But then $k - g_2 + g_3 = -g_2 + g_3 \notin G^+$. It follows that (RDP) does not hold.

**Example 3.7.** (RIP) $\not\Rightarrow$ ($\mathrm{RDP}_0$). Let $(E; +, 0, 1, )$ be the diamond (Dvurečenskij and Pulmannová, 2000, Example 1.9.23); that is, let $E = \{a, b, 0, 1\}$, and let $+$ be defined iff one argument is 0 or both arguments are $a$ or both are $b$, in which latter cases the sum is 1. Then $E$ is an effect algebra and a fortiori a pseudoeffect algebra.

As is easily checked, $E$ fulfils (RIP). But it does not fulfil ($\mathrm{RDP}_0$) , as is seen, for example, from the inequality $a \leq b + b$.

**Example 3.8.** An example of an effect algebra, so also of a pseudoeffect algebra, that does not fulfil (RIP) is the standard effect algebra $(\mathcal{E}(H); +, 0, I)$, where $\mathcal{E}(H)$ is the set of positive operators less than identity in an at least two-dimensional complex Hilbert space, $+$ is the sum of two operators defined when it is less than identity, and 0 is the zero, $I$ the identity operator.[2]

Let us first show that (RIP) fails when $H$ is two-dimensional, that is, $H = \mathbb{C}^2$. The positive operators of $H$ are those self-adjoint operators whose determinant and trace are $\geq 0$; so their matrices have the form

$$P = \begin{pmatrix} t - x & y - iz \\ y + iz & t + x \end{pmatrix}$$

for some $x, y, z, t \in \mathbb{R}$ such that $x^2 + y^2 + z^2 \leq t^2$ and $t \geq 0$. Let

$$A_1 = \frac{1}{18}\begin{pmatrix} 1 & 0 \\ 0 & 7 \end{pmatrix}, \quad A_2 = \frac{1}{18}\begin{pmatrix} 7 & 0 \\ 0 & 1 \end{pmatrix},$$

$$B_1 = \frac{1}{18}\begin{pmatrix} 9 & 4 \\ 4 & 9 \end{pmatrix}, \quad B_2 = \frac{1}{18}\begin{pmatrix} 9 & -4 \\ -4 & 9 \end{pmatrix}.$$

[2] We are indebted to Prof. Robin Hudson for his useful suggestions concerning the following proof.

Then we have $0 \leq A_1, A_2 \leq B_1, B_2 \leq I$. Suppose for some operator

$$C = \begin{pmatrix} t-x & y-iz \\ y+iz & t+x \end{pmatrix}$$

that $A_1, A_2 \leq C \leq B_1, B_2$. The conditions $C - A_1, C - A_2, B_1 - C, B_2 - C \geq 0$ lead to

$$\frac{2}{9} \leq t \leq \frac{1}{2},$$
$$\left(|x| + \frac{1}{6}\right)^2 + y^2 + z^2 \leq \left(t - \frac{2}{9}\right)^2,$$
$$x^2 + \left(|y| + \frac{2}{9}\right)^2 + z^2 \leq \left(\frac{1}{2} - t\right)^2,$$

from which we further derive $t \geq 7/18$ as well as $t \leq 5/18$. So (RIP) does not hold in the two-dimensional case.

Now suppose $H$ has any dimension $\geq 2$, and define the operators $A_1$, $A_2$, $B_1$, $B_2$, when restricted to some two-dimensional subspace $H_0$, as above, and let them map to 0 on $H_0^\perp$. Now, for another operator $C$, the condition $C \leq B_1$ means that also the kernel of $C$ includes $H_0^\perp$. So by the same reasoning as before, (RIP) fails to hold.

We note that $\mathcal{E}(H)$ is, like the other examples given above, an interval pseudoeffect algebra; we have $\mathcal{E}(H) = \Gamma(\mathcal{B}(H)_{\mathrm{sa}}, I)$, where $\mathcal{B}(H)_{\mathrm{sa}}$ is the set of all bounded self-adjoint operators of $H$.

A further property of Riesz type involves any finite number of elements rather than a maximum of four.

To make collections of formulas as they will appear in the sequel more easily readable, we use the following abbreviation. Let $d_{ij}, a_i, b_j \in E$, $1 \leq i \leq m$, $1 \leq j \leq n$. By

$$\begin{array}{ccccc} d_{11} & \cdots & d_{1n} & \rightarrow & a_1 \\ \vdots & & \vdots & & \vdots \\ d_{m1} & \cdots & d_{mn} & \rightarrow & a_m \\ \downarrow & & \downarrow & & \\ b_1 & \cdots & b_n & & \end{array}$$

we mean $d_{i1} + \cdots + d_{in} = a_i$ for $i = 1, \ldots, m$ and $d_{1j} + \cdots + d_{mj} = b_j$ for $j = 1, \ldots, n$.

**Lemma 3.9.** *Let* $(E; +, 0, 1)$ *be a pseudoeffect algebra fulfilling (RDP$_1$). Let*

$$a_1 + \cdots + a_m = b_1 + \cdots + b_n,$$

*where* $m, n \geq 1$. *Then there are elements* $d_{11}, \ldots, d_{mn} \in E$ *such that*

$$\begin{array}{cccccc} d_{11} & \cdots & d_{1n} & \rightarrow & a_1 \\ \vdots & & \vdots & & \vdots \\ d_{m1} & \cdots & d_{mn} & \rightarrow & a_m \\ \downarrow & & \downarrow & & \\ b_1 & \cdots & b_n & & \end{array}$$

*and such that for* $1 \leq i < m, 1 \leq j < n$, *we have*

$$d_{i+1,j} + \cdots + d_{mj} \,\mathbf{com}\, d_{i,j+1} + \cdots + d_{in}.$$

**Proof:** The lemma is trivial for $m = 1$ or $n = 1$, and it is true for $m = n = 2$ because of (RDP$_1$).

Suppose it is true for any pair of integers such that the first is less than or equal to $m$, where $m \geq 2$, and the second is less than $n$, where $n \geq 3$. From this condition, we shall prove that the lemma is also true for the pair $m, n$. By complete induction, it then follows that it is true in general.

So let $a_1 + \cdots + a_m = b_1 + \cdots + b_n$. By the induction hypothesis, there are elements $d_{11}, \ldots, d_{m,n-2}, e_1, \ldots, e_m \in E$ such that

$$\begin{array}{cccccc} d_{11} & \cdots & d_{1,n-2} & e_1 & \rightarrow & a_1 \\ \vdots & & \vdots & \vdots & & \vdots \\ d_{m1} & \cdots & d_{m,n-2} & e_m & \rightarrow & a_m \\ \downarrow & & \downarrow & \downarrow & & \\ b_1 & \cdots & b_{n-2} & b_{n-1} + b_n & & \end{array}$$

and for $1 \leq i < m, 1 \leq j \leq n-2$ we have

$$d_{i+1,j} + \cdots + d_{mj} \,\mathbf{com}\, d_{i,j+1} + \cdots + d_{i,n-2} + e_i.$$

Moreover, from $e_1 + \cdots + e_m = b_{n-1} + b_n$, we have by the same hypothesis that there are elements $d_{1,n-1}, d_{1n}, \ldots, d_{m,n-1}, d_{mn} \in E$ such that

$$\begin{array}{cccc} d_{1,n-1} & d_{1n} & \rightarrow & e_1 \\ \vdots & \vdots & & \vdots \\ d_{m,n-1} & d_{mn} & \rightarrow & e_m \\ \downarrow & \downarrow & & \\ b_{n-1} & b_n & & \end{array}$$

and for $1 \leq i \leq m-1$,

$$d_{i+1,n-1} + \cdots + d_{m,n-1} \,\mathbf{com}\, d_{in}.$$

It follows that the lemma is true for the pair $m, n$. □

The following consequences of the Weak Riesz Decomposition Property seem to be notable.

**Proposition 3.10.** *Every finite pseudoeffect algebra* $(E; +, 0, 1)$ *fulfilling* ($RDP_0$) *is commutative.*

**Proof:** Let $(E; +0, 1)$ be finite. Then every element $a \in E$ is the sum of finitely many atoms of $E$. Indeed, below every nonzero $a \in E$ lies an atom; otherwise, $E$ would not be finite. So choose an atom $e_1 \leq a$; if $a \neq e_1$, choose an atom $e_2$ below $e_1'$, where $e_1 + e_1' = a$; and so on. Then for some $n$, we have $a = e_1 + \cdots + e_n$; otherwise, $E$ would not be finite.

Now suppose that $E$ fulfils ($RDP_0$). Then any two atoms of $E$ commute. For either they are equal or their infimum is 0, whence, because of Lemma 3.2(ii), their sum exists and does not depend on the order. So we conclude that any two summable elements of $E$ commute. □

**Proposition 3.11.** *Every complete and atomic lattice pseudoeffect algebra* $(E; +, 0, 1)$ *fulfilling* ($RDP_0$) *is commutative.*

**Proof:** We claim that for every atom $a \in E$ and every $x \in E$, there is a largest $n \in \mathbb{N}$ such that $na \leq x$; we will denote this number by $j_a(x)$. For suppose there is no such $n$. Then $ma$ is defined for any $m$, so by Lemma 1.6(iv) and the completeness of $E$, we have $ma \leq a^-$ for any $m$, so $\bigvee_m ma \leq a^-$, which means that $(\bigvee_m ma) + a$ exists. By Remark 1.8, we conclude $(\bigvee_m ma) + a = \bigvee_m (ma + a) = \bigvee_m ma$, that is, $a = 0$.

It is now easy to see that for every $x \in E$, we may write $x = \bigvee_{a \in \mathcal{A}(E)} j_a(x)$, where $\mathcal{A}(E)$ is the set of all atoms of $E$.

As in the proof of Proposition 3.10, we see that multiples of two atoms commute. So by Remark 1.8, we have, for any $x, y \in E$,

$$x + y = \bigvee_{a \in \mathcal{A}(E)} j_a(x)a + \bigvee_{a \in \mathcal{A}(E)} j_a(y)a = \bigvee_{a \in \mathcal{A}(E)} \bigvee_{b \in \mathcal{A}(E)} (j_a(x)a + j_b(y)b) = y + x.$$

□

## 4. po-GROUPS WITH RIESZ PROPERTIES

We define the different properties of Riesz type for po-groups in exact analogy to those of pseudoeffect algebras.

*Definition 4.1.* Let $(G; +, \leq)$ be a directed po-group with neutral element 0.

(a) For $a, b, \geq 0$, we write $a$ **com** $b$ to mean that for all $a_1, b_1$ such that $0 \leq a_1 \leq a$ and $0 \leq b_1 \leq b$, $a_1$ and $b_1$ commute.
(b) We say that $G$ fulfils the *Riesz Interpolation Property* (RIP) if, for any $a_1, a_2, b_1, b_2$ such that $a_1, a_2 \leq b_1, b_2$, there is a $c \in G$ such that $a_1, a_2 \leq c \leq b_1, b_2$.
(c) We say that $G$ fulfils the *Weak Riesz Decomposition Property* ($\mathrm{RDP}_0$) if, for any $a, b_1, b_2 \geq 0$ such that $a \leq b_1 + b_2$, there are $d_1, d_2 \in G$ such that $0 \leq d_1 \leq b_1$, $0 \leq d_2 \leq b_2$, and $a = d_1 + d_2$.
(d) We say that $G$ fulfils the *Riesz Decomposition Property* (RDP) if, for any $a_1, a_2, b_1, b_2 \geq 0$ such that $a_1 + a_2 = b_1 + b_2$, there are $d_1, d_2, d_3, d_4 \geq 0$ such that $d_1 + d_2 = a_1$, $d_3 + d_4 = a_2$, $d_1 + d_3 = b_1$, and $d_2 + d_4 = b_2$.
(e) We say that $G$ fulfils the *Commutational Riesz Decomposition Property* ($\mathrm{RDP}_1$), if, for any $a_1, a_2, b_1, b_2 \geq 0$ such that $a_1 + a_2 = b_1 + b_2$, there are $d_1, d_2, d_3, d_4 \geq 0$ such that (i) $d_1 + d_2 = a_1$, $d_3 + d_4 = a_2$, $d_1 + d_3 = b_1$, $d_2 + d_4 = b_2$ and (ii) $d_2$ **com** $d_3$.
(f) We say that $G$ fulfils the *Strong Riesz Decomposition Property* ($\mathrm{RDP}_2$) if, for any $a_1, a_2, b_1, b_2 \geq 0$ such that $a_1 + a_2 = b_1 + b_2$, there are $d_1, d_2, d_3, d_4 \geq 0$ such that (i) $d_1 + d_2 = a_1$, $d_3 + d_4 = a_2$, $d_1 + d_3 = b_1$, $d_2 + d_4 = b_2$ and (ii) $d_2 \wedge d_3 = 0$.

**Proposition 4.2.** *Let $(G; +, \leq)$ be a directed po-group.*

(i) *We have the implications*

$$(RDP_2) \Rightarrow (RDP_1) \Rightarrow (RDP) \Rightarrow (RDP_0) \Leftrightarrow (RIP).$$

*There are no more implications holding between these conditions.*
(ii) *$G$ fulfils ($RDP_2$) if and only if $G$ is lattice-ordered.*
(iii) *Let $G$ be Abelian. Then we have the implications*

$$(RDP_2) \Rightarrow (RDP_1) \Leftrightarrow (RDP) \Leftrightarrow (RDP_0) \Leftrightarrow (RIP).$$

*The implication not shown does not hold.*

**Proof:** (ii) Let ($\mathrm{RDP}_2$) hold. Similarly as for Lemma 3.2(ii), we may prove that, for $a, b \in G$, from $a \wedge b = 0$, it follows that $a \vee b = a + b$. Furthermore, it is easy to see that, for $a, b, c \in G$, if $a \vee b$ exists, we have $c + (a \vee b) = (c + a) \vee (c + b)$. Now, we may, analogously to Proposition 3.3(ii), prove that, for $a, b \in G^+$, $a \vee b$ always exists.

Using Birkhoff (1995), XIII, §3, Eq. (7), we conclude that, for $a, b \in G^+$, $(a \vee b) - b = (a - b) \vee 0$ holds. Because $G$ is directed, we have $G = G^+ - G^+$ (Fuchs, 1963, Proposition II.1). So, for all $g \in G$, $g \vee 0$ exists, which, by Birkhoff (1995), XIII, §3, Lemma 1, means that $G$ is an $\ell$-group.

Conversely, in all $\ell$-groups, ($RDP_2$) holds (Fuchs, 1963, Theorem V.1).

(i) If ($RDP_2$) holds, $G$ is, by (ii), an $\ell$-group. Elements $a$ and $b$ in an $\ell$-group are such that $a \wedge b = 0$ commute (Birkhoff, 1995, XIII, §3, Eq. (13)); so ($RDP_1$) holds. From ($RDP_1$) follows (RDP), and from (RDP) follows ($RDP_0$) trivially. ($RDP_0$) and (RIP) are equivalent according to Fuchs (1965), Theorem 2.2.

That the converse implications in three cases do not hold is seen from the examples given above for pseudoeffect algebras. So the group of rational funtions from [0, 1] to $\mathbb{R}$ with no singularities (see Example 3.4) fulfils ($RDP_1$), but not ($RDP_2$). The group described in Example 3.5 fulfils (RDP), but not ($RDP_1$). The group described in Example 3.6 fulfils (RIP), but not (RDP).

(iii) ($RDP_1$) trivially follows from (RDP). (RDP) is to be derived from ($RDP_0$) in the obvious manner (Goodearl, 1986, Proposition 2.1). The other implications are proved in (i).

Example 3.4 shows that from ($RDP_1$), property ($RDP_2$) does not follow. □

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